The Geometry and Game Theory of Chases

1. Introduction

In this paper, we will investigate the hunting strategies of predators and the fleeing strategies of their prey in a chase of finite time. In particular, we will consider the case of a predator-prey relationship which might have taken place 75 million years ago, during the late Cretaceous period: we shall suppose that the velociraptor (*Velociraptor mongoliensis*) is hunting the thescelosaurus (*Thescelosaurus neglectus*). Clearly, if one of the two "players" in this deadly game is both faster and more agile than the other, it has an overwhelming advantage. In this particular predator-prey relationship, no such overwhelming advantage exists; the velociraptor is faster, but the thescelosaurus is more agile, with a minimum turning radius much smaller than that of the velociraptor. Our goal is to find the optimal strategies for both the hunter and the hunted, in the case of a single velociraptor hunting a single thescelosaurus as well as that of a pair of velociraptors hunting a single thescelosaurus.

This original problem actually has a low probability of having occurred, as fossil remains of the velociraptor have been found only in Mongolia, while fossil remains of the thescelosaurus have been found only in the midwestern region of the United States and Canada (Weishampel et al. 270, 500). However, this model can be useful in the study of a wide range of such problems, simply by varying the parameters. In studying these particular creatures, we may come to understand the trade-off between speed and maneuverability.

2. Parameters, Assumptions, Preliminary Calculations

•We are given as parameters that the velociraptor moves with a speed of v_v= 60 k/h, or 16.7 m/s, and the thescelosaurus moves with a speed of v_t = 50 k/h, or 13.9 m/s. We are also given that the velociraptor has a hip height of 0.5m. It is estimated that an velociraptor's turning radius is three times its hip height; thus the velociraptor can turn with a minimum radius r_v = 1.5 m. Moreover, we are given a thescelosaurus minimum turning radius of r_t = 0.5 m. We assume further that they always find it more to their advantage to turn with this wide radius than to decelerate, come to a stop, change direction, and accelerate once more. (This makes sense because most animals, such as dogs and squirrels, turn rather than stop short and change direction, at least in our experience.)

•The game in question has a fixed maximum length. In general, most hunts are limited by some finite time, e.g. the maximum endurance (or patience) of the predator, or the onset of night or day. In this particular case, it is limited by the pitiful endurance of the otherwise fearsome velociraptor. After a burst of speed of time T = 15 s, we are given that the velociraptor must stop to rest, while the thescelosaurus can run for a comparatively infinite length of time. Here, we make the additional assumption that the velociraptor must rest for more than a time $\left(\frac{v_v - v_t}{v_t}\right)T = 3$ s, i.e. more than the time required for the thescelosaurus to run as far as the maximum distance the velociraptor could close in 15 s. Thus the velociraptor must catch the thescelosaurus in the first 15 seconds after its presence becomes known.

•The thescelosaurus will be able to detect the approach of a velociraptor when the distance between the two is less than some length D; we are given 15 m $\leq D \leq$ 50 m. We assume, moreover, that as the velociraptor is stalking the thescelosaurus, and not the other way about; thus the velociraptor will be able to detect its prey farther than 50 m away. Furthermore, for the second part of the problem, we assume that D is not dependent upon angle; i.e., if each member of a pair of velociraptors approaches from a different angle relative to the thescelosaurus, the thescelosaurus will detect each one when its distance from the thescelosaurus is less than D.

•We will assume that due to the position of the eyes on opposite sides of the dinosaurs' heads, their vision is virtually 360°; thus they are aware of the position of their opponent independent of orientation.

•The average human reaction time is ≈ 0.1 s; the thescelosaurus can turn around 180° in this amount of time. We will assume that animals with the speed and agility of these dinosaurs have a considerably smaller reaction time Δt, which we will vary between 0.005 s and 0.05 s in our study. Furthermore, we will assume that when the thescelosaurus has to register the presence of two velociraptors instead of one, this additional burden on its senses does not actually change its reaction time.

•From a picture of the velociraptor (Czerkas 28 compared with Paul 363), we deduce approximate measurements: a body length of 3 m, foreclaw length 0.5 m, hip-to-foreclaw distance 0.6 m, and hip-to-head distance 1.2 m. Moreover, a running bipedal dinosaur, due to its long tail, has its center of gravity close to the hips (Alexander 69). Based on these measurements of the bipedal velociraptor, we assume that, while running at top speed, it will catch anything that comes within a distance $\delta_v = 0.6$ m of its position, which we define to be the place on its torso from which the foreclaws extend. At the widest point of its torso, the velociraptor is only 0.4 m wide, and thus we can ignore this thickness as it is contained well within the δ_v reach of its arms. Note that the location of the center of this reach is not at the hips, which is the point from which we assume the turning radius was calculated by the scientists. However, we shall see that this slight incongruity does not qualititatively change our approach to the problem.

•The thescelosaurus is given to be a biped of similar size. For the velociraptor to catch the thescelosaurus,we assume it must be able to grab at the torso, as the head and tail are too thin to easily grab at 60 km/h. Thus we will represent the thescelosaurus as a circle of radius $\delta_t = 0.2$ m over its hips. If the grabbing region of the velociraptor intersects this circle, the thescelosaurus has been caught.

•In order to facilitate calculation we will assume that both predator and prey move at full speed for the entire time of the hunt *T*, even when they are moving in curves, with radii of curvature no less than r_v and r_t, respectively. Note that this assumption is not entirely reasonable, as one can calculate the centripetal acceleration to be 19*g* and 39.4*g* respectively (*g* being gravitational acceleration at the surface of the earth) . Given time for further investigation, it would be appropriate to model the dinosaurs with a maximum acceleration up to a top speed.

•For the second part of the problem, with two velociraptors, we will assume that the velociraptors work perfectly together:

a) a velociraptor has just as much incentive to let its companion catch the thescelosaurus as to catch the prey itself.

b) the velociraptors are perfectly coordinated and can communicate their plans.

c) the velociraptors must allow each other space to move; we will assume that this is equivalent to preventing their grabbing regions from intersecting.

3. Analysis: One Velociraptor

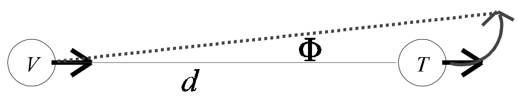
3.1 Approach

We will begin by considering the simple case of the velociraptor initially chasing the thescelosaurus on a straight line, separated by a distance *d* significantly larger than the turning radii of the dinosaurs, as shown at below.

The thescelosaurus's goal is to evade the velociraptor for however much time *T-t* remains before the velociraptor runs out of endurance. Thus, if $d > (v_v - v_t)(T\text{-}t)$, the thescelosaurus can run directly away from the velociraptor, and the velociraptor cannot close the distance in the time remaining.

But what if $d < (v_v - v_t)(T\text{-}t)$? (Certainly this will be the case if the velociraptor can approach the thescelosaurus undetected to a distance closer than $(v_v - v_t)T = 42$ m.) In this scenario, we know that the thescelosaurus must make use of its superior maneuverability if it is to survive, while the velociraptor's primary goal is to close the distance between itself and the velociraptor.

For sufficiently large *d*, no matter how the thescelosaurus turns, it is a trivial matter for the velociraptor to adjust its course to keep heading directly toward its prey. Consider the diagram below; note the small angle Φ by which the velociraptor must adjust its course. Thus while *d* is large, the velociraptor can adjust its course appropriately to directly close in on the thescelosaurus.

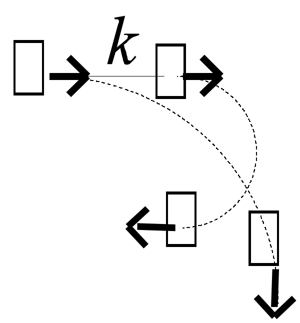


3.2 Encounter

The thescelosaurus must now make some decisions: When has the velociraptor come near enough for the thescelosaurus to make use of its superior agility (while not getting eaten), and how should it let the velociraptor approach? We will consider two representative strategies.

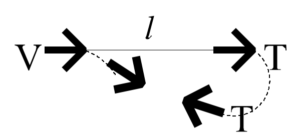
Encounter A

In this strategy, the thescelosaur will initially run directly away from the velociraptor. This costs the velociraptor time as it can only close the relative distance at a rate of $v_v - v_t$. Once the velociraptor has closed to within a distance *k*, the thescelosaurus will use its superior maneuvaribility to "dive" out of the way. In the diagram at right, the thescelosaurus will turn at its minimum turning radius; the velociraptor will turn at its maximum turning radius to intercept, but it is too late. The distance *k* must be chosen with great care; if it is too large, the velociraptor will be able to adjust its angle and close on the thescelosaur. If it is too small, the thescelosaur will not be able to get out of the way of the velociraptor's grabbing radius.

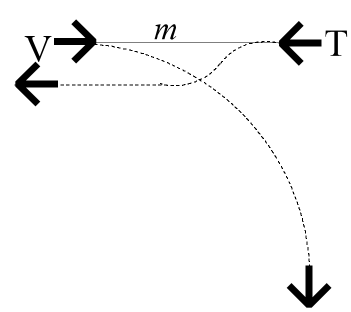


Encounter B

In this strategy, the thescelosaurus will allow the velociraptor to only close to a distance *l*, considerably greater than the distance *k* in strategy A.. Upon reaching this point, the the thescelosaurus will then turn around and head directly *toward* the velociraptor. (See ① at right.) The velociraptor, of course, will continue to close; the distance between them will now shrink at a rate $v_v + v_t$. At an appropriate distance *m*, the thescelosaur will again dive out of the way. (See ② at right.) In comparison to strategy A, however, the thescelosaur will be even more successful at dodging the velociraptor, as it need only change its course by a small amount and it will fly by the velociraptor at a relative velocity of approximately



①



②

$v_v + v_t$. The value for *m* must be chosen with great care: if it is too small, the thescelosaur will not be able to stay outside the reach of the velociraptor, while if it is too large, the velociraptor will be able to compensate, and intercept the thescelosaurus.

For a given set of parameters (speeds, turning radii, grabbing radius of the velociraptor), there may or may not be values of *k*, *l*, or *m* such that strategies A and/or B will allow the thescelosaurus to survive. We propose to study this property of the parameters.

3.3 Endgame

If the thescelosaurus survives the encounter, the velociraptor will attempt to turn around, and once again close in its prey. The thescelosaurus will have one of two strategies in this situation:

Endgame A

Run away! If the distance between the thescelosaur and the velociraptor is now greater than $(v_v - v_t)(T\text{-}t)$, this is clearly the best approach; it will escape unscathed as the velociraptor runs out of endurance.

Endgame B

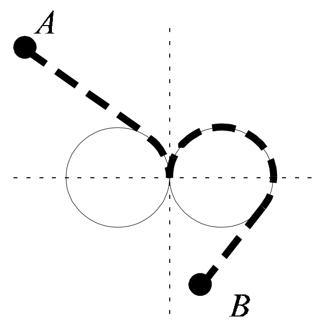
This is a slightly more daring maneuver, but will take a big chunk of the velociraptor's time. Instead of running away from the velociraptor, the thescelosaurus should try to curve around it, ending up *directly behind* the velociraptor. The velociraptor must turn around to come at the thescelosaurus; due to its superior agility, the thescelosaur may be able to remain in this position relative to the velociraptor for some time. If the velociraptor starts to turn left, the thescelosaur will also start to turn left, attempting to remain 180° behind it. Due to its superior speed, however, the velociraptor will eventually outdistance the thescelosaurus, and the thescelosaurus will no longer be able to stay directly behind it. At this point, the thescelosaur should resort to Endgame A, as the velociraptor will soon turn around and chase it.

At the end of this post-encounter "endgame," the velociraptor will once again be chasing the thescelosaurus, and we return to the "approach" phase.

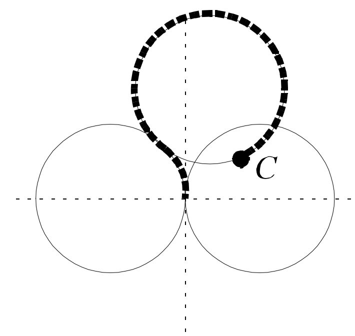
4. Modeling the Chase: One Velociraptor

4.1 The Velociraptor Metric

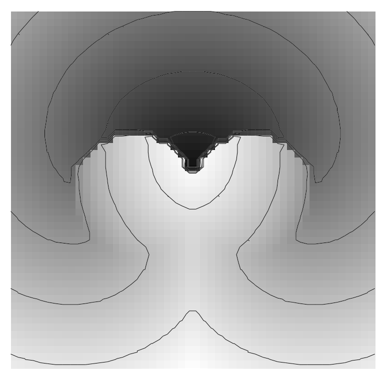
We will begin our model of the chase by asking the following question: How would the velociraptor get from point A to point B? More precisely: If the velociraptor is at the origin of the plane, facing in the positive *y*-direction, how would it get to the point (x, y). Since we are considering the velociraptor to have constant speed, it should simply take the shortest path from the origin to the point. Unfortunately for the velociraptor, this distance is not the Euclidean metric, since the velociraptor has a limited turning radius! It cannot take a straight line path. So what is the appropriate path?



In the diagram at right, we have placed the velociraptor at the origin, and the two circles to either side represent the path of minimum turning radius. For points (such as the points *A* and *B*) outside these circles, the choice of minimum distance path is fairly clear. The velociraptor will turn around the circle of minimum radius until it is heading directly toward the destination point. It will then leave the circle and head straight toward the point. Representative paths are shown at right as dashed lines. Note that it is always advantageous to turn toward the side of the plane on which the destination point lies. (For the calculation of this length, refer to the Appendix.)

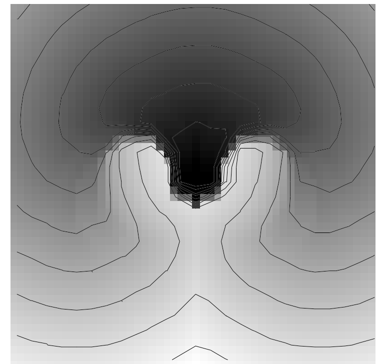


Points *inside* the circles of minimum turning radius are more difficult for the velociraptor to access. It must somehow move such that destinations points inside these circles (e.g. point *C*) are on or outside the circles of minimum curvature. The shortest way to do this is to turn *away* (in this case, to the left) from the destination point, following the other circle of minimum radius. Once the destination point lies outside the circles of minimum radius, follow the circles to the point, as shown at right. (For the calculation of this length, refer to the Appendix.)

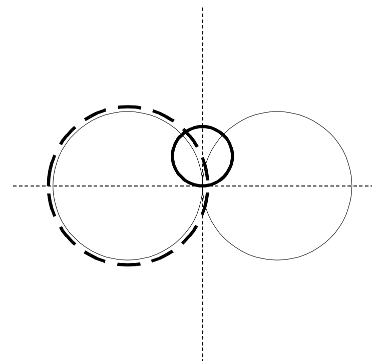


We now define a new metric (metric 1) on the plane: the distance along the curve from the origin (the location of the velociraptor, with the velociraptor facing the positive *y*-direction) to the given point that the velociraptor will follow. This metric is represented at right as a density plot with contour lines superimposed. Darker regions correspond to shorter distances for the velociraptor. The circles of minimum turning radius are easy to see due to the discontinuity of the metric on the portions of the circles above the *x*-axis. (Plots generated using Mathematica.)

In the calculation of our metric thus far, we have simply considered the velociraptor to be a point. But in truth, the velociraptor has a grabbing radius of δ_v! Thus to access a given point in theplane it only needs to reach any point a distance δ_v from it. We will assume that the velociraptor



will chose to go to the point within a distance δ_v from the destination that minimizes the distance it must travel. Therefore, we replace the value of the metric at each destination point with the minimum of the value of the original metric on a disk of radius δ_v surrounding the desination point, yielding the plot at right (Metric 2). Note that the only parameters on which metrics 1 and 2 depend are the grabbing radius and minimum turning radius of the velociraptor, and metric 1 is simply metric 2 assuming a grabbing radius of zero.



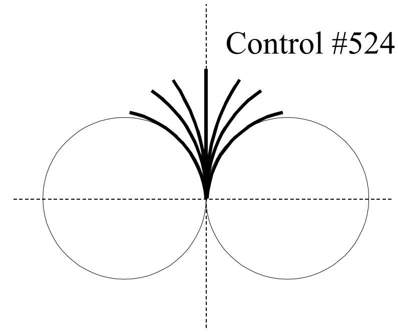
Now we will treat a subtlety we alluded to in the previous section. As mentioned in the assumptions, the origin of this coordinate system (the velociraptor's center of gravity) is actually 0.6 m *behind* the center of the grabbing radius. However, one can see from the diagram at right that given the circular and straight-line motions discussed above, the model of the situation remains exactly the same if we shift the origin to the center of the velociraptor-ball and simply change the effective minimum turning radius to $\sqrt{1.5^2+0.6^2} = 1.6$ m.

We can further simplify our model in the following manner: we know that the velociraptor has caught the thescelosaurus if their effective regions (circles of radius δ_v and δ_t, respectively) overlap. This is equivalent to saying that the centers of the two circles are separated by a distance less than $\delta = \delta_v+\delta_t$, or that the velociraptor has a grabbing radius of δ, and the thescelosaur is a point. We thus define our metrics as described above, with effective turning radius 1.6 and effective grabbing radius $\delta = \delta_v+\delta_t = 0.8$ m.

4.2 Dinosaurs have brains the size of a peanut

Given the velociraptor metric, we will first assume that the velociraptor will simply act to minimize the value of the metric and the thescelosaurus will act to maximize it. We will see later that this is not sufficient to examine all possible strategies (in particular, Encounter strategy B, wherein the thescelosaurus heads straight for the velociraptor until the last possible moment), and we will soon make our dinosaurs a bit more sophistocated. However, we begin with this as a first approximation.

To evolve the system in time at a given time t, each dinosaur will consider the location and heading of its opponent. It will then choose how to move during the next time step Δt. (Note that the value of Δt is approximately equivalent to its reaction time, as it is only after the time Δt that the dinosaur will next be able to evaluate the movement of its opponent.) As a range of options, the dinosaur will choose from a selection of arcs with length $v\Delta t$ and radii between the minimum turning radius and infinity (a line segment), shown at right, below. To choose among them, the dinosaur may choose the path with the most advantageous endpoint. Or, it may extrapolate each path several more timesteps, and choose from among these based on the metric evaluated at their endpoints. We will vary this choice of strategy in our analysis of the model.



Using Borland C++, we developed a computer simulation of the dinosaurs' behavior. Using this strategy (wherein the velociraptor always attempts to minimize the metric and the thescelosaurus always attempts to maximize), several phenomena discussed in section 3 were observed. (Perhaps most importantly, the dinosaurs chose paths similar to those used in determining the metric.)

When the dinosaurs were separated by a distance larger than approximately three meters, we observed the "approach" phase of the chase. The thescelosaurus would run directly away from the velociraptor, while the velociraptor would adjust its course to trail directly behind, closing the distance. Under most circumstances, the thescelosaurus would attempt to "shake" the velociraptor, but since the velociraptor was a sufficient distance behind, it was easy for it to adjust its course appropriately. Thus we observed a rapid (on the order of a time-step ≈ 0.01 s), small amplitude oscillation of the thescelosaurus's direction in the approach phase. Once the velociraptor gets close enough to the thescelosaurus in the simulation, the thescelosaurus will adopt encounter strategy A (discussed in section 3.2). If it survives, it adopts one of the endgame strategies. In Color Plates 1 and 2, we show a hunt wherein the thescelosaurus successfully evades the velociraptor for fifteen seconds using encounter strategy A followed by endgame strategy B.

We further found that the thescelosaurus performed better using metric 2 looking only one temp-step ahead. Metric 2 is clearly advantageous for the prey because this metric teaches it to stay out of the path of the predator's grabbing radius, rather than simply avoiding its center. The thescelosaur relies on its ability to maneuver quickly; thus it constantly adjusts its heading, rendering it useless for it to estimate several time-steps into the future.

The velociraptor performed optimally using metric 1, looking five time-steps ahead. We originally programmed the velociraptor to use metric 2, but it turned out to be a bit too cocky; the velociraptor was constantly dissapointed as the thescelosaurus barely slipped out of reach. When we changed its strategy to employ metric 1, this problem was eliminated. In a future investigation, it may be useful to make the velociraptor to use metric 2 with a non-zero grabbing radius smaller than the actual value.

We now ask what parameters allow the thescelosaurus to survive. For the given speeds and minimum turning radius, the thescelosaurus will always survive for values of the effective grabbing radius $\delta < 0.4$ m, and is always captured for $\delta > 0.5$. In the region in between, the outcome is highly sensitive to initial conditions. Unfortunately for the thescelosaur, the given value of δ is actually 0.8 m. Thus, the thescelosaur should try encounter strategy B.

4.3 The thescelosaurus learns to play "chicken"

We now wish to consider encounter strategy B. However, as strategy B requires the thescelosaurus to head directly toward the velociraptor, this is clearly incompatible with the thescelosaurus looking forward a few time to see which path will maximimize its distance from the velociraptor (based on metric 2). Thus, we must modify our simulation to study this strategy.

After the encounters described in 4.2 in which the thescelosaurus escapes, the thescelosaurus is usually able to increase the distance between itself and the velociraptor to ≈ 7.5 m, and the next encounter occurs ≈3.2 s later. We will assume that this is sufficient time and distance for the thescelosaurus turn around such that it is heading directly toward the velociraptor when the encounter begins. We therefore will only consider part 2 of encounter strategy B, wherein the thescelosaurus dives out of the path ot the velociraptor just before collision.

We assume that once the thescelosaurus begins to dodge, it is simply resuming its original strategy of maximizing the distance (according to metric 2) between the two dinosaurs (See color plate 3). Thus, we can simulate this strategy using the original simulation, with the initial condition that the dinosaurs are heading straight toward each other separated by a small distance *m*, as shown in ② of the diagram of encounter strategy B in section 3.2.

We found that if the thescelosaurus runs towards the velociraptor, and starts turning when he is 2.15 m from the velociraptor, he will escape every time from a grabbing radius of 0.6, even if the other parameters are changed slightly. Thus strategy B is a clear improbement over strategy A.

A rough calculation using the numbers we know reassures us that such a strategy will indeed work: after one such pass, the thescelosaurus has a 7 m lead, and in the time it takes for the thescelosaurus to turn around 180 degrees the velociraptor will gain about 1.9 m; this leaves the thescelosaurus a bit of maneuvering before the critical 2.15 m turning point.

4.4 The velociraptor takes a gamble, or, **The rational dinosaurs**

If the parameters are such that the thescelosaurus can escape using strategy B, what is the velociraptor to do? In our study of this strategy, we found that for a grabbing radius of 0.6 m, the 2.15 m critical distance had very little room for error--if the thescelosaur dodges too early or too late by 0.1 m, it will be caught. Thus, the velociraptor knows exactly when the thescelosaurus will make its dodge.

If the velociraptor pursues its until-now optimal strategy of minimizing his metric, as above, it will lose its dinner every time. Therefore, it is to its advantage to try to anticipate the movement of the thescelosaurus; if the velociraptor guesses correctly which way the thescelosaurus will swerve, and correct its own course accordingly, it can gain valuable time and thus catch its prey. However, if it does not guess correctly, it will lose even more time than if it had merely gone straight. Moreover, the thescelosaurus pursued by such a decision-oriented velociraptor, for its part, also wants to anticipate the movement of its predator.

We can model this as a game theory problem. Consider the last possible moment before the thescelosaurus must swerve. Under our original strategy B, the thescelosaurus will swerve either left or right either left or tight at this time. The velociraptor, knowing this, should arbitrarily choose to swerve either left or right in this instance, giving it a 50% chance of guessing correctly and catching its prey. However, the thescelosaurus knows this! So, if it is sure that the velociraptor will swerve, it should keep going straight *past* the critical point, and once the velociraptor swerves it can dodge the the other way an instant later. The velociraptor, knowing this, realizes it is not always to its advantage to anticipate the movement of the thescelosaur; perhaps the thescelosaur will anticipate its anticipation. In this situation, the velociraptor's optimal strategy is to keep going straight! The thescelosaur, then past the critical point, will be eaten.

Thus, it may be reasonable for the thescelosaur to move left (L), right (R), or stay straight toward the center of the velociraptor (C). The velociraptor can choose to antipate these moves; we thus denote the velociraptor's strategy by L, C, or R. If the velociraptor's guess is correct, we assyne it will catch the thescelosaur, receiving a normalized payoff of 1, and the thescelosaur receives a payoff of zero. If the velociraptor guesses incorrectly, the thescelosaurus will survive the encounter, and the game will be played again at the next encounter, and so on until the velociraptor's endurance runs out.

If the thescelosaurus swerves one way and the velociraptor anticipates the other (a "large miss"), then it will be a decent interval of time before the velociraptor catches up to the thescelosaurus for the next encounter. If, however, one of the dinosaurs goes straight and the other swerves (a "small miss"), it will take less time for the velociraptor to catch up. Thus, there will be fewer encounters for the remainder of the hunt following a large miss than there will be after a small miss, and thus the probability p that the thescelosaurus survives the hunt after a small miss is less than the probability q that the thescelosaurus survives the hunt after a large miss. In the small miss case the thescelosaur's payoff is therefore p, and that of the velociraptor is 1-p. In the large miss case, the thescelosaur's payoff is q, and that of the velociraptor is 1-q. These results are sumarized at right.

		Thescelosaurus		
		L	C	R
Velociraptor	L	1,0	1-q,q	1-p,p
	C	1-q,q	1,0	1-q,q
	R	1-p,p	1-q,q	1,0

In this analysis, we have made several assumptions. In particular, we have simplified the pay-offs such that all small misses result in the same payoffs, and all large misses result in the same payoffs. This may not be entirely correct, as small misses come in two different forms: those in which the velociraptor goes straight, and those in which the thescelosaurus goes straight. We have also assumed that the dinosaurs are symmetrice, and do not prefer one side to the other.

We would like to find any Nash equilibrium of this payoff matrix. It is clear that there is no pure eqilibrium, so we look for a mixed strategy. Let a and b be the probabilities the velociraptor and thescelosaurus choose L, respectively. Since there is no difference between choosing right or left, we can say that the probability a dinosaur picks L is equal to the probability that it picks R. Thus the probability that the dinosaurs choose strategy C is 1-2a and 1-2b, respectively.

Now finding the mixed Nash equilibrium is pedestrian. As in the elementary game-theory problem, each dinosaur wants to maximize its own expected payoff, and minimize that of the other. This will occur when the expected payoffs of the opponent are equal for any of its strategies.

Let $P_t(V \blacktriangleright L)$ be the expected payoff for the thescelosaur if the velociraptor chooses L. Thus we have $P_t(V \blacktriangleright L) = P_t(V \blacktriangleright R) = (1 - 2b)q + bp$, and $P_t(V \blacktriangleright C) = 2qb$. Setting these thescelosaurus payoffs equal, we find $b = \frac{q}{4q - p}$.

Similarly, we have $P_v(T \blacktriangleright L) = P_v(T \blacktriangleright R) = a + (1 - q)(1 - 2a) + a(1 - p)$ and $P_v(T \blacktriangleright C) = 2a(1 - q)+(1 - 2a)$. Setting these equal, we find that a is also $\frac{q}{4q - p}$.

Thus, to determine the probabilities a and b, we must determine p and q. If there is only time remaining in the chase for one encounter, then $p = q = 1$, as the thescelosaur will not have another chance. Thus $a = b = 1/3$, and thus each dinosaur will choose each of its three strategies with equal probablilities, and the thescelosaur will escape 2/3 of the time! Thus if both dinosaurs know that only one encounter remains, their expected payoffs from that encounter are 1/3 (for the velociraptor) and 2/3 (for the thescelosaurus).

Now suppose (for example) that in a given encounter, if there is a small miss there will be time for one more encounter, while there will not be time for another encounter if there is a large miss. Thus $p = 1$, since a large miss means the thescelosaur survives the chase, and $q = 2/3$, since the probablility of the thescelosaur surviving the next encounter is 2/3, from the previous paragaph. Therefore, $a = b = 2/5$.

Given more time for this study, time-dependent values of p and q could be determined, and thus we could determin the subgame-perfect Nash equilibrium of the dinosaurs for the entire chase.

4.5 Additional Comments

We also observe (refer to sensitivity section) that the simulator running encounter strategy A is highly sensitive to small changes in various parameters (most strikingly, reaction time), whereas encounter strategy B is not. Given this sensitivity along with the bigger escape range of strategy B, we see that the optimal evasion strategy for the thescelosaurus is to use strategy B in two regions: a) the region where the grabbing radius is sufficiently small such that strategy B will always work, as strategy A will not always work everywhere in this region; b) the region where strategy B sometimes works but strategy A never works.

5. Two velociraptors: what changes?

5a. Approach

As before, it is to the thescelosaurus's advantage to run directly away from the velociraptors when the distance between them is large. With two velociraptors, this translates to the thescelosaurus running such that the distance between it and one velociraptor remains the same as the distance between it and the other velociraptor. In this large-distance limit, the strategy for the velociraptors is also clear: they should run towards the thescelosaurus using the same strategies as above.

There is one substantial difference between this case and that of one velociraptor: at the very beginning of the chase, the initial configuration is specified not only by D (which suffices for the single velociraptor and prey), but also by the angle made by the two velociraptors with the thescelosaurus as the vertex. It is to the velociraptors' advantage to start out 180° apart, assuming the thescelosaurus moves in a straight line and the velociraptors continually adjust their directions to intercept it (a reasonable assumption in the far distance limit). This arrangement produces the maximum possible initial approach velocity: the velociraptor's maximum velocity v_v.

5b. Encounter

By the time the velociraptors close in on the thescelosaurus such that the prey will will have to start to curve, the configuration of the two velociraptors plus thescelosaurus approaches one of only two cases. In the first case, both velociraptors run neck-to-neck and hence act roughly as one velociraptor with a rather large turning radius. In the second case, the velociraptors and the thescelosaurus form a straight line, but one velociraptor is behind another.

There are, then, two main strategies for the velociraptors: either to run neck-to-neck or for one to run behind the other and pounce as soon as the thescelosaurus starts turning. The neck-to-neck strategy is quite easy to model; as the two predators act as one, this case is a simple variant of the one-predator case. As one might expect, the critical radius turns out to be half that of the one-predator case. This model is probably fairly biologically accurate for radii larger than the critical radii, as few calculations and complex maneuverings are required. This strategy should be chosen for the appropriate critical-radius case

The consecutive-velociraptor strategy, on the other hand, is a bit trickier. This strategy is meant for very small critical radii (0.3 or less). The idea is for one velociraptor to “corral” the thescelosaurus by curving *toward* it, even though according to the distance metric this actually makes the thescelosaurus farther away. This maneuver thus restricts the movement of the thescelosaurus by a great deal; the other velociraptor, which has been cruising behind the corraling velociraptor during this maneuever, can then circle in for the kill.

Preliminary studies using the simulator show that this strategy shows a great deal of merit. However, the simulator does break down; the would-be corralling velociraptor curves the wrong way. Although we did not program the simulator to allow the animals to work together as the preceding paragraph implies, we are confident that this strategy will work for small radii if such experiments are done. Note also that it becomes much harder for the thescelosaurus to dive between the predators, although this is probably possible for small enough critical radius.

6. Sensitivity of the model

We tested the model by varying the parameters of the simulator, most notably the reaction time and the grabbing radius. Changing the curving radii and the relative velocities, while undoubtedly having a pronounced effect on the outcome, does not exhibit counterintuitive, extremely sensitive, or chaotic behavior.

Varying the reaction time led to interesting results. It turns out that the model is rather sensitive to the reaction time (though not unpredictable); over a variation of 0.04 s in reaction time, the critical radius could shrink by as much as 0.1 m, not a trivial factor when the critical radius is only about 0.4. Interestingly enough, the velociraptor does best when its reaction time is relatively long (0.05 s), whereas the thescelosaurus does best when its reaction time is relatively short (0.01 s). Therefore, we dealt with this sensitivity by choosing the reaction time for each dinosaur as that where it did best (where closest approach distance was maximized, for the thescelosaurus; and minimized, for the velociraptor), all other parameters (including the reaction time of the other dinosaur) held constant.

As mentioned before, strategy A is extremely sensitive in the 0.4-0.5 region to changes in initial position and velocity. Not only is this behavior sensitive, but it also seems to be chaotic. Such unpredictable behavior is a weakness in strategy A. Strategy B exhibits no such sensitivity; thus the sensitivity does not become a weakness of the model overall.

7. Strengths and weaknesses of the model

Our model has many strengths, perhaps the greatest of which is that the model is easy to understand: minimization and maximization of a metric is a simple concept, and one which is not hard to implement.

Another prime strength of our model is the extreme robustness of the simulator. Not only can the simulator handle a wide range of similar scenarios simply by changing the parameters involved, but it can also handle a variety of different strategies simply by adjusting the initial conditions accordingly, as we did with strategy B. However, we were not able to reprogram the simulator to deal with the cooperation between two predators. Also, the dependence upon sensitivity is somewhat troublesome, though not debilitating.

Moreover, we feel that that the part of our model incorporating strategy B has many virtues to recommend it. Its robustness, as discussed in the previous section, lends it credibility as a feasible strategy. In addition, its applicability to a relatively large subset of turning radii makes it a better strategy than any other we could find. Furthermore, the game theory presented can be applied to most such finite games. However, we have a caveat in that in an actual situation involving live creatures, the prey would almost certainly not have the presence of mind to realize that running straight towards its predator would be the optimum strategy.

One of the main weaknesses of our model is the assumption that the dinosaurs go at top speed even when they are turning. More realistically, they should slow down to go around the curves at a reasonable centripetal acceleration. As we mentioned in our analysis of the problem, we have made progress on changing the metric to take this deceleration into account, but we were unable due to time constraint to fully explore this aspect of the model. However, we do present thoughts on this direction for further analysis of the problem below.

8. Addendum: Realism rears its ugly head

As we noted in the Assumptions section, the assumption that the dinosaurs are going at their top speeds even on curves is a rather poor one due to the tremendous centripetal accelerations that would be involved. A better approximation is to model the dinosaurs' velocity on a circle arc as related to the radius of that circle. Since for centripetal acceleration, $a = v^2/r$, a first approximation is to assume that the dinosaurs can sustain an acceleration of, say, $3g$, and can keep that maximum acceleration no matter what the radius of curvature is. Then we can model the velocity as a function of the radius as $v(r) = \min(\sqrt{a/r},\ v_{max})$.

The natural question we must ask is this: does this change the optimal strategy for the velociraptor?

A second approximation is to take into account the tangential acceleration and deceleration to the curved paths when the radius of curvature is changing; this not completely negligible, since if the

9. References

Alexander, R. McNeill. Dynamics of Dinosaurs and Other Extinct Giants. New York: Columbia University Press, 1989.

Czerkas, Sylvia J., and Olson, Everett C, ed. Dinosaurs Past and Present, vol. II. Seattle: University of Washington Press, 1987.

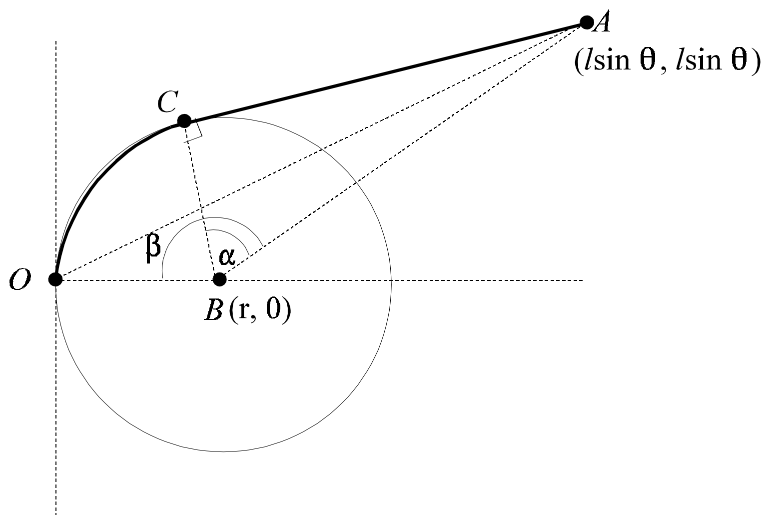
Paul, Gregory S. Predatory Dinosaurs of the World. New York: Simon and Schuster, 1988.

Weishampel, David B.; Dodson, Peter; and Osmolska, Halszka. The Dinosauria. Los Angeles: University of California Press, 1990.

Appendix: Calculation of the velocirapator metric

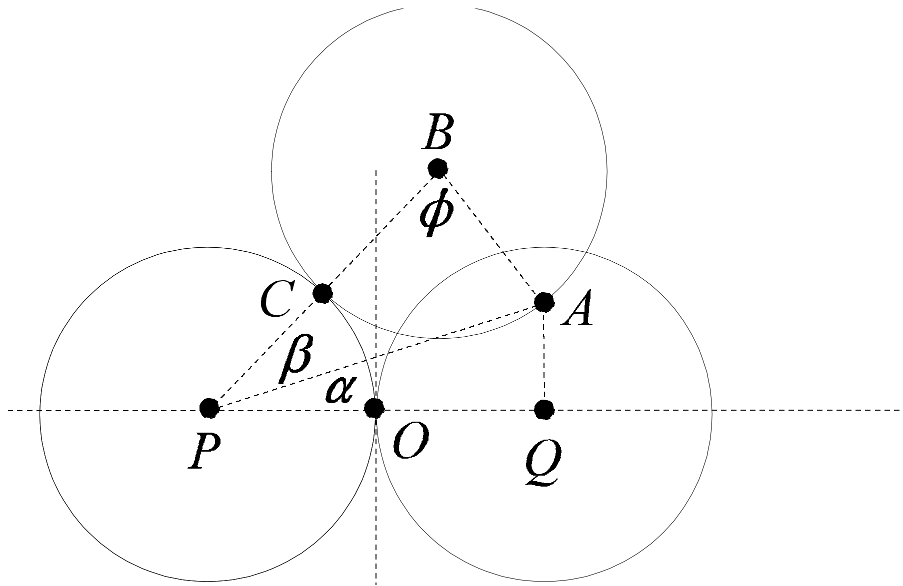
1. Outside the circles of minimum turning radii

Let the velociraptor (currently considered to be a point) be located at the origin, facing in the positive *y*-direction. Let the destination point be *A*, a distance *l* from the velociraptor, at an angle θ from the velociraptor's heading; thus, $A = (l\sin\theta, l\cos\theta)$. We will abbreviate the minimum turning radius $\mathbf{r}_v$ as *r*. Let $B = (0, r)$ be the center of the circle of minimum turning radius, and let *C* be the point at which the velociraptor leaves the circle and moves along a straight line to point *A*. (Thus line *AC* is tangent to the circle.) Let $\alpha = \angle ABC$, and $\beta = \angle OBC$. This is summarized in the diagram below:



The total distance the velociraptor must travel is the length *AC* and the length of the arc *OC*, which is $r(\beta - \alpha)$. From the given coordinates, we have $AC = \sqrt{(l\sin\theta - r)^2 + (l\cos\theta)^2}$, and $BC = r$, so the length $AC = \sqrt{(l\sin\theta - r)^2 + (l\cos\theta)^2 - r^2} = \sqrt{l(l - 2r\sin\theta)}$. The law of cosines tells us that given a triangle with sides *a*, *b*, and *c*, the angle opposite the side with length *c* is $\cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$. We have found *AB* above; we know $OB = r$, and $OA = l$; thus we have angle β. The angle α is one of the acute angles in right triangle *ABC*, thus we can say that $\alpha = \tan^{-1}\left(\frac{AC}{BC}\right)$. Thus, the total length is $\sqrt{l(l - 2r\sin\theta)} + r\cos^{-1}\frac{r - l\sin\theta}{\sqrt{l^2 - 2lr\sin\theta + r^2}} - r\tan^{-1}\frac{\sqrt{l(l - 2r\sin\theta)}}{r}$.

2. Inside the circles of minimum turning radii



Let A be the destination point, P and Q the centers of the circles of minimum turning radius, and B the center of the right-hand circle of minimum turning radius after the velociraptor moves through arc OC, as shown above. Let angles ϕ, α, β, be as shown above. The velociraptor must move through minor arc OC and major arc AC, for a total length of $(2\pi - \phi + \alpha + \beta)$. Let A have coordinates (x, y). The Pythagorean theorem yields that $AP = \sqrt{(x + r)^2 + y^2}$, and $AQ = \sqrt{(x - r)^2 + y^2}$. The lengths PB, PQ, and AB are $2r$, $2r$, and r, respectively. Thus we may apply the law of cosines as described in A.1, and find the angles in question.