Week 1: Vectors, Dot Products, and Cross Products

A few key points

- A vector is a quantity with a direction and length (magnitude)
- If $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and likewise for $\vec{b}$, then

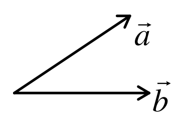
  Length (by Pythagorean theorem): $|\vec{a}| = \sqrt{a_x^{\,2} + a_y^{\,2} + a_z^{\,2}}$

  $\vec{i}$, $\vec{j}$, and $\vec{k}$ components add separately: $\vec{a} \pm \vec{b} = (a_x \pm b_x)\vec{i} + (a_y \pm b_y)\vec{j} + (a_z \pm b_z)\vec{k}$
- Comparison of dot and cross products:

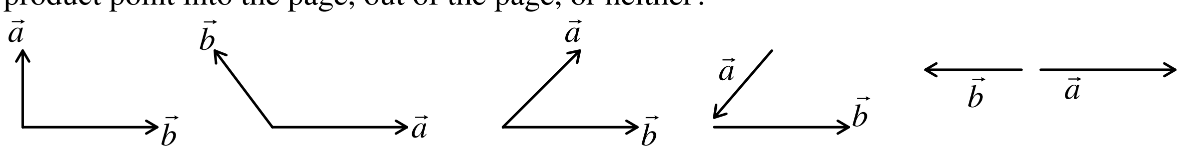
Dot product: $\vec{a}\cdot\vec{b}$	Cross product: $\vec{a}\times\vec{b}$
scalar (number)	vector
$\vec{a}\cdot\vec{b} = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\cos\theta$	$\vec{a}\times\vec{b} = (\lvert\vec{a}\rvert\lvert\vec{b}\rvert\sin\theta)\vec{n}$
	$\vec{n}$ perpendicular to $\vec{a}$ and $\vec{b}$, determined by right hand rule
Commutative: $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$	Anti-commutative: $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$
Distributive: $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$	Distributive: $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

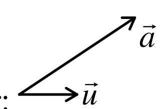
- By definition, $\vec{i}$, $\vec{j}$, and $\vec{k}$ form a right-handed coordinate system
- Dot product useful for: finding amount of force in a particular direction, finding how much light goes through polarizers
- Cross product useful for: calculating torque

Problems



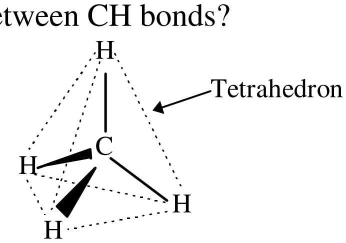
1. Draw $2\vec{a}+\vec{b}$, $\vec{a}+\frac{3}{2}\vec{b}$, and $\frac{3}{2}\vec{a}-\vec{b}$. Would drawing a grid help?
2. For each of the following pairs of vectors: a) is the dot product positive, negative, or 0; b) does the cross product point into the page, out of the page, or neither?



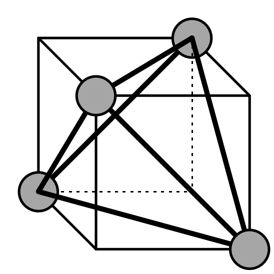


3. In the following diagram, draw a length representing $\vec{a}\cdot\vec{u}$ if $\vec{u}$ is a unit vector:
4. $\vec{a}$, $\vec{b}$, and $\vec{c}$ all lie in the same plane. Find $(\vec{a}\times\vec{b})\cdot\vec{c}$.
5. Section 10.3, problem 16
6. Methane (CH_4) has a central carbon bonded to 4 hydrogens at the vertices of a tetrahedron. What is the angle between CH bonds?

Methane



Hint: A tetrahedron can be formed from alternate corners of a cube.



7. Simplify: $\vec{a}\cdot(\vec{b} - proj_{\vec{a}}\vec{b})$. Hint: Draw a picture instead of writing down equations.

Week 2: Lines, Planes, and Vector functions

A few key points

- Equations for lines and planes (think about **why** these equations graph as lines or planes)
  Line through (x_0, y_0, z_0) parallel to $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$: $x=x_0+at$, $y=y_0+bt$, $z=z_0+ct$ (parametric equation)
  Plane through (x_0, y_0, z_0) with normal vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ (this equation is equivalent to setting the dot product of the normal vector and a vector in the plane equal to 0)
- Distance from lines and planes to a point always refers to the closest distance (equivalently, the perpendicular distance).
  Distance from a line to a point is $|PS \times \hat{v}|$, where P is a point on the line, S is the point, and $\hat{v}$ is a unit vector pointing along the line.
  Distance from a plane to a point is $|PS \cdot \hat{n}|$, where P is a point on the plane, S is the point, and $\hat{n}$ is a unit normal vector to the plane.
  How do you know which one involves the dot product and which one involves the cross product?
- To differentiate or integrate vector-valued functions, simply differentiate or integrate each component separately.
- Arc length of a vector-value function between t=a and t=b is $\int_a^b |\vec{v}(t)|dt$. Analogously, how do you find the distance traveled by a particle in one dimension given the velocity as a function of time?

Problems

1. Section 10.4, problem 30ab.

2. Section 10.5, problem 72.

3. Section 11.1, problem 40.

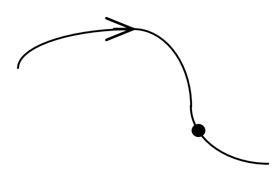
4. Do the following lines intersect? If so, where? If these equations described the trajectories of particles, do they collide? If so, where?
   Line 1: $x=1+2t$, $y=2+t$, $z=3-t$ Line 2: $x=3-5t$, $y=3-t$, $z=2+2t$

5. Show that for a point moving around a circle at a constant velocity, the magnitude of the acceleration is v^2/r, where v is the velocity of the point, and r is the radius of the circle. Why does the acceleration always point towards the center of the circle? Where would the acceleration vector point if the point's velocity changed as it went around the circle?

6. Plot this vector-valued function. Is it smooth? What is its derivative?
   $\mathbf{r}(t) = \sin(t)\,\mathbf{i} + \sin(t)\,\mathbf{j}$

7. You rotate the vector $x\,\mathbf{i} + y\,\mathbf{j}$ counterclockwise in the xy plane by an angle θ. Where does it end up?

8. The figure to the right shows the graph of $\mathbf{r}(t)$ oriented in the direction of the arrow. Draw in a vector representing the direction of $\mathbf{r}'(t)$ at the dot. Will $\mathbf{r}''(t)$ point to the right or left of $\mathbf{r}'(t)$?

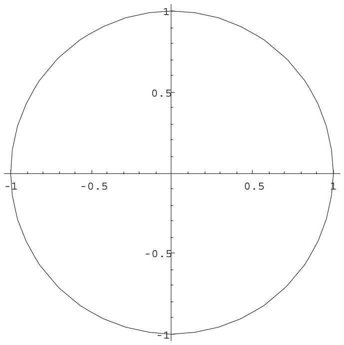


CA: Ed Boas

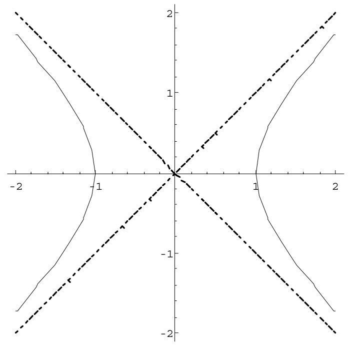
Week 3: Quadratic Surfaces

A Visual Dictionary of Quadratic Curves and Surfaces

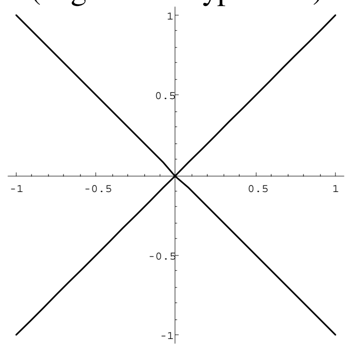
$x^2+y^2=1$
Circle



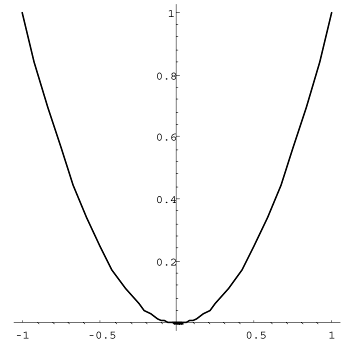
$x^2-y^2=1$
Hyperbola



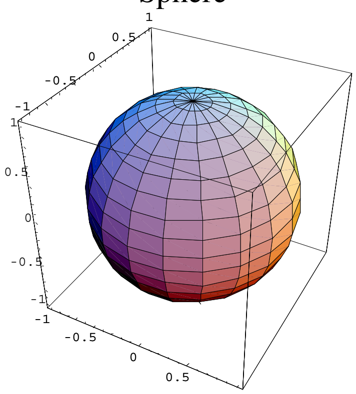
$x^2-y^2=0$
Intersecting lines
(degenerate hyperbola)



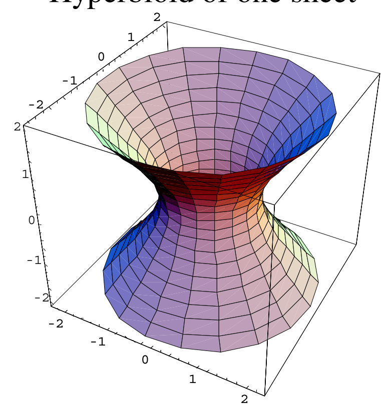
$y=x^2$
Parabola



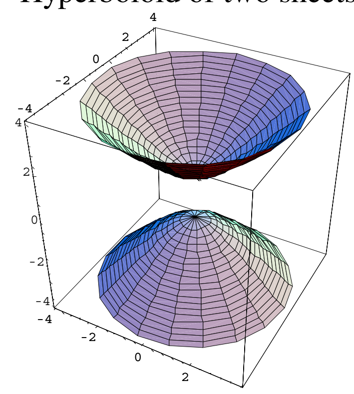
$x^2+y^2+z^2=1$ (No negative coeff.)
Sphere



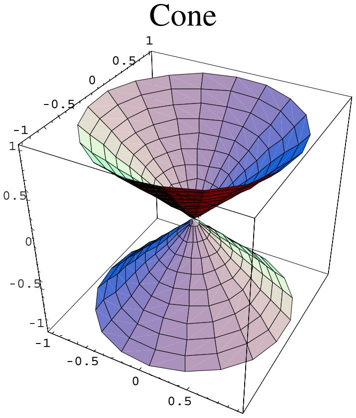
$x^2+y^2-z^2=1$ (One negative coeff.)
Hyperbloid of one sheet



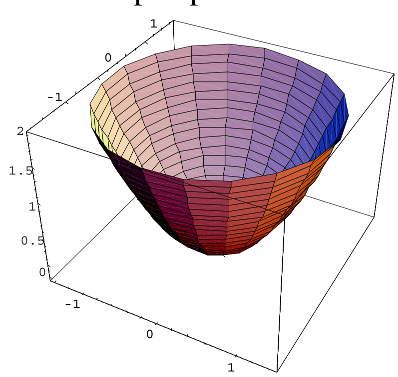
$-x^2-y^2+z^2=1$ (Two negative coeff.)
Hyperboloid of two sheets



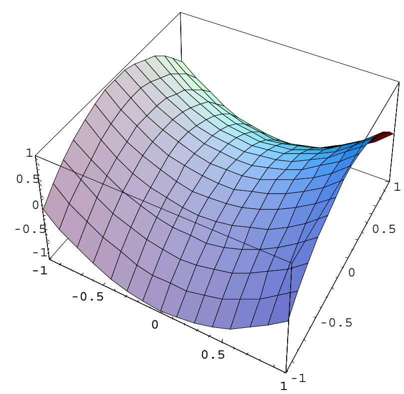
$-x^2-y^2+z^2=0$
Cone



$z=x^2+y^2$
Elliptic paraboloid



$z=x^2-y^2$
Hyperbolic paraboloid



- The graph of $f(x-x_0, y-y_0, z-z_0)=0$ is the graph of $f(x, y, z)=0$ with the origin translated to (x_0, y_0, z_0)
- The graph of $f(x/a, y/b, z/c)=0$ is the graph of $f(x, y, z)=0$ stretched out by a in the x direction, etc.

Problem

Is the graph of $xy=1$ a hyperbola? If so, can you do a change of variables (i.e. change your coordinate system) to make the equation look like the equation for a hyperbola given above?

Week 4: Partial derivatives

A few key points

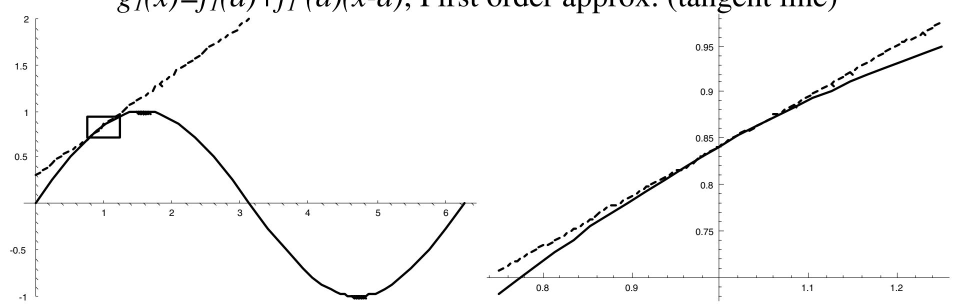
- **Derivatives** — When computing a derivative, keep track of whether the letters in your equation stand for constants, variables, or functions. For example, $\frac{d}{dx}(x^2) = 2x$ because x is a variable, $\frac{d}{dx}(y^2) = 0$ if y does not depend on x, and $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ by the chain rule if y is a function of x.
- **Partial derivatives** — Order of multiple derivatives doesn't matter for smooth functions. For example, $f_{xy}=f_{yx}$..
- **Taylor series** — The point of Taylor series is to approximate $f(x)$ near a with a polynomial $g(x)$ whose derivatives at $x=a$ match those of $f(x)$:

  $g(x)= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \ldots$
- **Approximations** — The Taylor series often converges to the exact value of $f(x)$; to obtain an approximation, just sum a few terms and report the next term as the "error." This is shown below for sin x.

Function —— $f_1(x)=\sin x$

Approx. ······ $g_1(x)=f_1(a)+f_1'(a)(x-a)$; First order approx. (tangent line)

Graphs

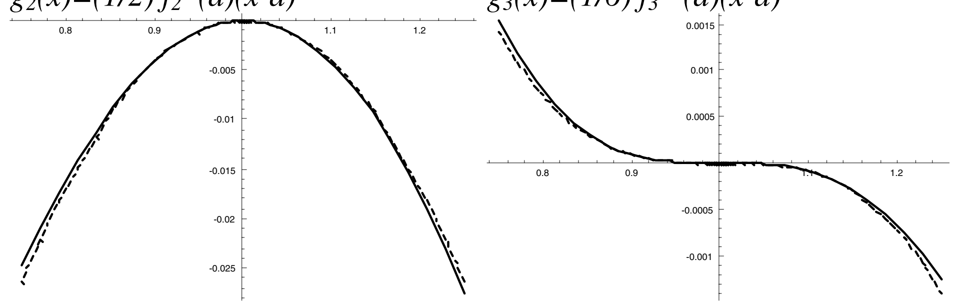


The tangent line approximation gets better as you "zoom in" on sin x.

Error in previous approx. —— $f_2(x)=f_1(x)-g_1(x)$ $f_3(x)=f_2(x)-g_2(x)$

Approx to error. ······ $g_2(x)=(1/2)\,f_2''(a)(x-a)^2$ $g_3(x)=(1/6)\,f_3'''(a)(x-a)^3$

Graphs



Left graph: The error in the tangent line approx. to sin x looks like a parabola (solid line) and is approximated by the $(x-a)^2$ term of the Taylor series (dotted line).
Right graph: The error in the parabolic approximation in the left graph looks like a cubic (solid line) and is approximated by the $(x-a)^3$ term of the Taylor series (dotted line)
Note the changes in the y scale of the graphs reflecting the fact that each subsequent term of the Taylor approximation (usually) adds a smaller and smaller correction.

★**Linear approximations, Multivariable Taylor series, Tangent planes, Chain rule, Gradients, Directional derivatives** — These are just different ways of stating the *same* concept: Over small intervals, the value of some function can be **approximated by a linear equation**. This linear equation comprises the linear terms in the **Taylor series**. For a function of two variables,

$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \ldots$ This is also the equation of the **tangent plane** to *f* at (*a*,*b*). By rearranging this formula we get the **chain rule** (applicable when x and y are functions of another variable t): $\frac{f(x,y)-f(a,b)}{\Delta t} = \frac{f_x \Delta x + f_y \Delta y}{\Delta t} \xrightarrow{\text{take limit of small } \Delta t} \frac{d}{dt} f(x,y) = f_x x_t + f_y y_t$. We can rearrange the chain rule in terms of the **gradient**:
$\frac{d}{dt} f(x,y) = f_x x_t + f_y y_t = (f_x \vec{i} + f_y \vec{j}) \cdot (x_t \vec{i} + y_t \vec{j}) = (\nabla f) \cdot \vec{v}$, where $\vec{v}$ is the velocity of the point (x, y). If $\vec{v}$ is a unit vector, then $(\nabla f) \cdot \vec{v}$ is the **directional derivative** of f in the direction $\vec{v}$. Thus, *df/dt* is maximum when $\vec{v}$ is parallel to ∇f, so the direction of the gradient is the direction in which *f* is "steepest," and a vector perpendicular to the gradient tells is the direction for which *df/dt*=0, i.e. the direction of a contour line.

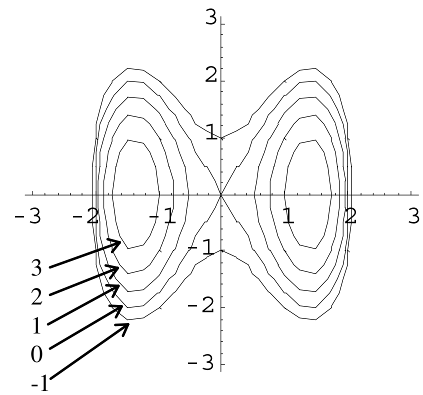
Problems

This week's material

1. For $z=x^2+y^2$, find:
   A. Taylor series approximation to z near x=1, y=2.
   B. Use (A) to approximate $(1.01)^2+(1.98)^2$
   C. Tangent plane and normal vector at x=1, y=2.
   D. If *dx*/*dt*=-1, *dy*/*dt*= -1, will *dz*/*dt* be positive, negative, or zero? Compute *dz*/*dt*.
   E. grad z
   F. On a graph of $z=x^2+y^2$, what is the direction of steepest positive slope? Steepest negative slope? 0 slope?
2. Consider $w(x,y,z)=x^2+y^2+z^2$.
   A. Find *dw/dz*.
   B. For what values of (x,y,z) is *dw/dz* positive? Negative? Zero? Why?
   C. What do the level surfaces of *w* look like?
   D. Express *w* in spherical coordinates.
   E. In spherical coordinates, what is $dw/d\phi$? $dw/d\theta$? $dw/d\rho$?
3. Boston is at 42°N latitude, 71°W longitude. The Aukland, New Zealand is at 42°S, 173°E. The radius of the earth is 4000 mi. What is the distance between the two cities (on the sphere)? [Hint: Use the dot product]
4. The volume of a can with radius *r* and height *h* is $V(r,h) = \pi r^2 h$. The surface area of the can is $S(r,h) = 2\pi r^2 + 2\pi r h$. Are *dS*/*dr*, *dS*/*dh*, *dV*/*dr*, and *dV*/*dh* positive, negative, or zero?
5. In polar coordinates, $x = r \sin\theta$ and $y = r\cos\theta$. What are dx/dr, $dx/d\theta$, dy/dr, $dy/d\theta$?
6. In polar coordinates, why is this incorrect: $r = x/\sin\theta \Rightarrow dr/dx = 1/\sin\theta$? [Hint: Is θ a function of x?]
7. In polar coordinates, what are dr/dx, $d\theta/dx$, dr/dy, $d\theta/dy$?
8. What is the arc length formula for the polar graph ($r(t)$, $\theta(t)$)? [Hint: Start with the arc length formula for parametric equations in Cartesian coordinates]

Midterm review

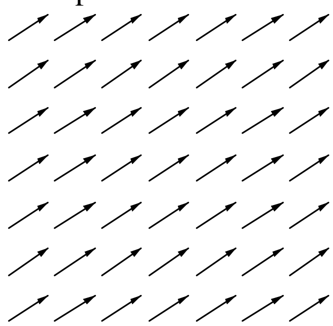
1. Problem 11.additional.3 (velocity of rotating vectors)
2. Problem 10.additional.12 (geometry with vectors)
3. The graph to the right shows contour lines for f(x,y). Value of each contour line indicated by arrows.
   A. Approximate: $f(-1,1)$, $f_x(-1,1)$, $f_y(-1,1)$, grad $f\,|_{(-1,1)}$.
   B. Plot $f(x,0)$.
4. Find a parametric equation for the line of intersection of $x+2y+z=0$ and $3x+2y-z=0$.



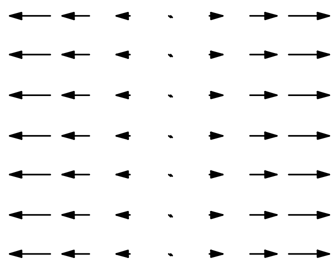
Week 5: Gradients, Vector Fields, and Line Integrals

A few key points

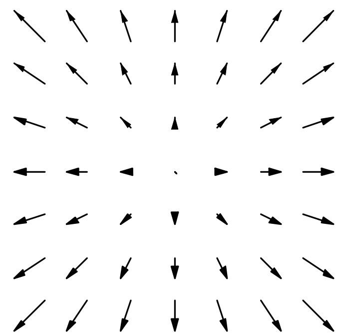
Examples of vector fields



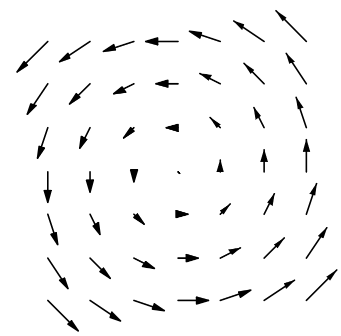
F(x,y)=3**i**+2**j**



F(x,y)=x**i**



F(x,y)=x**i**+y**j**



F(x,y)= -y**i**+x**j**

- **Line integral** — A “normal” integral such as $\int_a^b f(x)dx$ means to add up the values of *f(x)* times *dx*, an infinitesimally small increment in *x*, as *x* varies from *a* to *b*. A line integral such as $\int_C \vec{F}(x,y)\cdot d\vec{r}$ means to add up all the values of $\vec{F}(x,y)\cdot d\vec{r}$ as (*x*, *y*) trace out a curve *C*. $\vec{F}(x,y)$ is a vector field like the ones shown above and $d\vec{r}$ is an infinitesimal vector pointing along the path. The usual way to evaluate the line integral is to express it in the form $\int_C \vec{F}(x,y)\cdot\vec{v}dt$. $\vec{F}(x,y)\cdot\vec{v}$ is a scalar, which you should compute as a function of time, turning this a “normal” integral.
- Review the gradient.
- **Fundamental theorem of line integrals** — These two statements are equivalent:
  1. A vector field $\vec{F}(x,y,z)$ is the gradient of a scalar function f(x,y,z)
  2. Any line integral over $\vec{F}$ from (x_0, y_0, z_0) to (x_1, y_1, z_1) will have the value f(x_1, y_1, z_1)-f(x_0, y_0, z_0) (i.e. the integral is path independent).

  [Note: in physics terms, $\vec{F}$ is a conservative force (such as gravity, springs), and f is the potential]
- **Why is the fundamental theorem useful?** — Instead of computing a line integral for work, just subtract the potential energies at the endpoints!

Potential
(Scalar function)

gradient →

Conservative force field
(vector field)

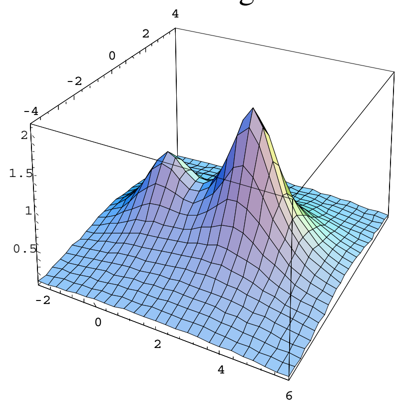
← “inverse” gradient
equivalently, line integral from a fixed point

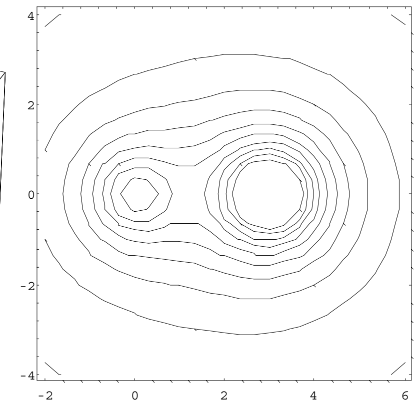
- Potential is not defined for a force field which is not conservative because line integrals will depend on the path

Problems

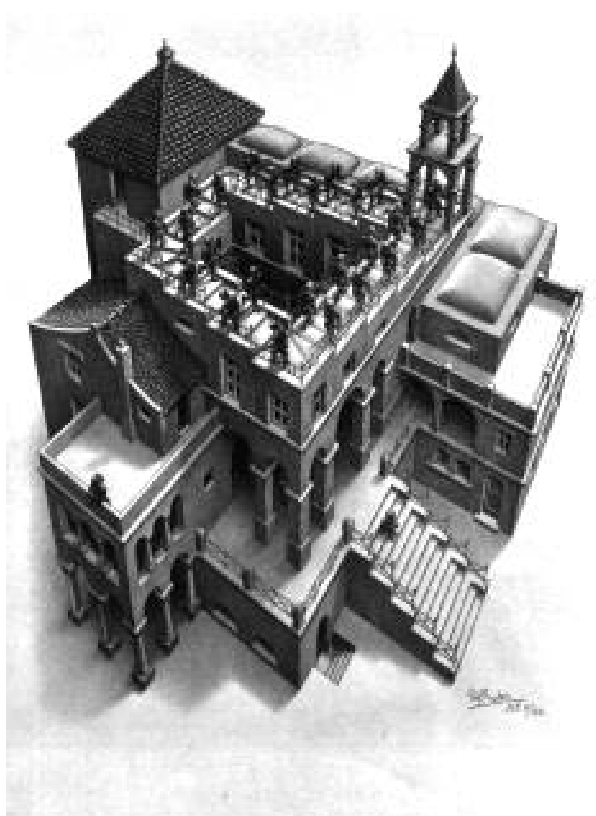
1. Find the arc length formula for polar coordinates.
2. Problem 12.5.40 (Box with changing dimensions)
3. Problem 10.7.20 (Converting to cylindrical and spherical coordinates)
4. Midterm.7 (Planetary motion-like question)

5. For each of the vector fields shown above, find the line integral over the paths:
   A. Line from (0,0) to (2,1)
   B. Line from (0,0) to (0,1) then to (2,1)
   C. Parabola $y=x^2/4$ from (0,0) to (2,1)
   D. Counterclockwise circle with radius 1 centered about the origin.
6. A 3D and a contour plot (with evenly spaced contours) of a hill is shown to the right.





   A. On the contour plot, draw a few vectors representing the gradient.
   B. Does the gradient point uphill or downhill?
   C. Parallel to contour lines or perpendicular?
   D. How does the length of the gradient depend on the spacing of the contours?
   E. What is the integral of the gradient around a closed loop?
   F. The mountain peak is 1 unit tall. What line integrals of the gradient also equal 1?
7. Why isn't $\mathbf{F}(x,y)= -y\mathbf{i}+x\mathbf{j}$ conservative? Does it have anything to do with the following Escher print? Can you walk to school in 6 feet of snow uphill both ways?

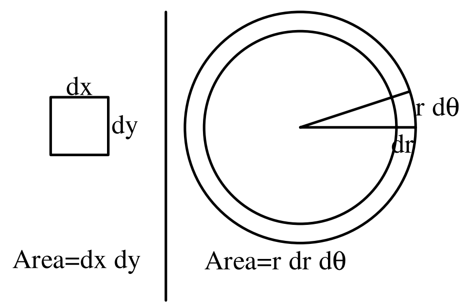


8. Find potential functions for the vector fields pictured at the top of the first page. Which one(s) don't have a potential function? Why?

Week 6: Multiple Integrals

A few key points

- **Single variable integrals** — Just a few reminders
  - To integrate $\cos^2 x$, use the identity $\cos^2 x=(1+\cos 2x)/2$
  - To integrate $\sin^2 x \cos x$, use u-substitution with $u=\sin x$
  - To integrate xe^{-x^2}, use u-substitution with $u=x^2$
  - To integrate $x \cos x$, integrate by parts (twice)
  - There is no chain rule or product rule for integrals!
- **To find the limits of an integral** — choose the outer limits first, or you will get extra variables in your answer
- **To set up an integral** — remember that an integral is just a sum. For example:
  - The volume of a box is the area of the base times the height. To find the **volume under a surface** $z=f(x,y)$, break it into boxes with infinitesimally small bases. Then, for each box, the area of the base is $dx\,dy$, the volume is $f(x,y)\,dx\,dy$, so the total volume under the surface is $\int\int f(x,y)dxdy$.
  - Moment of inertia of a point mass is $I=mr^2$, where m is the mass and r is the distance from the axis. To find the **moment of inertia** of some complex shape with density $\rho(x,y,z)$, break it up into infinitesimal pieces. Then, for each piece, $dx\,dy\,dz$ is the volume, $\rho(x,y,z)\,dx\,dy\,dz$ is the mass, $r^2\rho(x,y,z)\,dx\,dy\,dz$ is the moment of inertia, so $\int\int\int r^2\rho(x,y,z)dxdydz$ is the total moment of inertia.
  - The x coordinate of the center of mass of some shape with density $\rho(x,y)$ is just a weighted average of the x coordinates. The "weight" is proportional to the mass, and the "weights" must add up to 1 since this is an average. For each infinitesimal piece of the object, $\rho(x,y)\,dx\,dy$ is the mass, $\rho(x,y)\,dx\,dy$ / (total mass) is the "weight," so $\int\int x\rho(x,y)dxdy$ / (total mass) is the x coordinate of the center of mass.
- Integrals in polar coordinates — ($dx\,dy$ = an infinitesimal area $dA = r\,dr\,d\theta$)

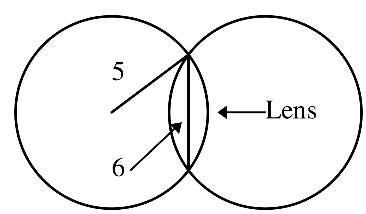


Problems

1. Find $\int_0^1\int_0^2\int_0^3 dxdydz$. What might this be the volume of?
2. A. Evaluate $\int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dxdy$ by first converting to polar coordinates.

   B. Factor $\int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dxdy$ into the product of two single variable integrals.

   C. Using (A) and (B), find $\int_0^\infty e^{-x^2}\,dx$



3. What is the volume of a biconvex circular lens with diameter 6 units whose surface is made from a sphere with radius 5? A side view of the lens is shown to the right. Express the volume as a both a double and triple integral.
4. What is the mass of the lens from problem 3 if the density is 1+|z|/2, where the z axis is perpendicular to the plane of the lens's circular edge?
5. The volume element $dV = dx\,dy\,dz$ in rectangular coordinates. What is dV in cylindrical coordinates?
6. Find the moment of inertia of a disk of radius r and mass m about an axis perpendicular to the disk.
7. Find the moment of inertia of a solid sphere of radius r and mass m about a diameter.

Week 7: Multiple Integrals and Optimization

A few key points

- **The parts of a multiple integral** are shown to the right.

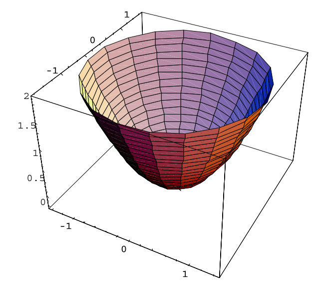
$$\underbrace{\int_a^b \int_{c(z)}^{d(z)} \int_{e(y,z)}^{f(y,z)}}_{\text{The limits of integration determine the shape of the region being integrated}} \underbrace{g(x,y,z)}_{\text{Function being integrated}} \underbrace{dx\,dy\,dz}_{\text{Infinitesimal volume or area element}}$$

- **The region of integration** includes all points (x, y, z) such that $x>a$ and $x<b$ etc. In other words, the region includes everything within the limits, not just the boundary. Also note that the limits can only be functions of variables that are integrated outside of it.
- **The infinitesimal volume element** dV or the area element dA depends on the coordinate system. Thus,

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

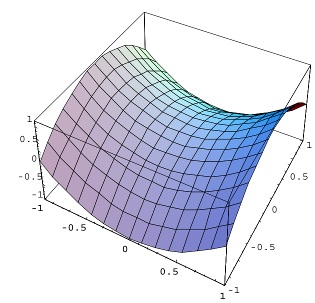
$$dA = dx\,dy = r\,dr\,d\theta$$

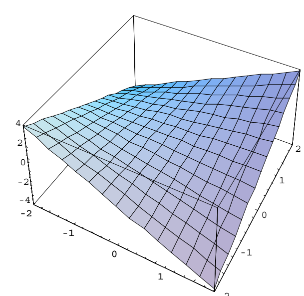
- **To maximize or minimize a function of several variables**:
  1. Find the critical points [points for which all the single variable derivatives are 0 or undefined]
  2. For each critical point, compute the discriminant $D \equiv f_{xx} f_{yy} - f_{xy}{}^2$
     If D>0, then the critical point is a max or min; use either f_{xx} or f_{yy} to determine which
     If D<0, then the critical point is a saddle point
  3. Check the boundary points (for example, by parameterizing the boundary and plugging into your function to turn it into a one variable optimization problem)



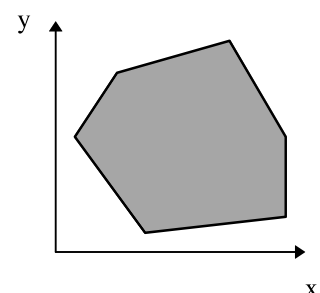
Problems

1. Set up integrals to find the area of a circle, volume of a cylinder, volume of a sphere, and area of a rectangle using the most appropriate coordinate system.
2. Problem 13.6.12b (find volume of an ice cream cone)
3. Using the fact that the area of a circle is πr^2, integrate $\int_0^1 2\sqrt{1-x^2}\,dx$.
4. Using the fact that the volume of a sphere is $\frac{4}{3}\pi r^3$, what is the volume of the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2=1$?
5. Evaluate $\int \frac{r^2}{1+r^2}\,dr$ (an intermediate step in one of the homework problems)
6. For each of the graphs to the right, find the signs of f_x, f_y, f_{xx}, f_{yy}, f_{xy}, D at the origin. Is the origin a critical point? A max? A min? A saddle point? [Note: the rightmost graph is the middle graph rotated by 45° about the z axis]





7. Find the dimensions of a cylinder which has the highest volume / surface area ratio.
8. You wish to maximize the function f(x, y) = (ax + by) inside the polygonal region shown to the right. Which (x, y) could be extrema? Corners? Edges? Interior points?



9. For what (x,y) is the line x=1+t, y=2-t closest to the origin. Solve with and without Lagrange multipliers.
10. Find m and b to make $y=mx+b$ a least-squares regression line through a set of points (x_i, y_i). In other words, minimize $\sum_i (mx_i + b - y_i)^2$. [Hint: Set this up like a normal optimization problem and treat x_i and y_i like constants.]

Week 8: Second Hourly Review

Problems

Optimization

1. Find the dimensions of a cylinder with the highest volume / surface area ratio.
2. For what (x, y) is the line x=1+t, y=2-t closest to the origin? Solve this using vectors and by setting up an optimization question.
3. A circle of radius 5 is centered at (1,2). What points of the circle are closest and furthest from the origin? Solve both geometrically and by setting up an optimization question.

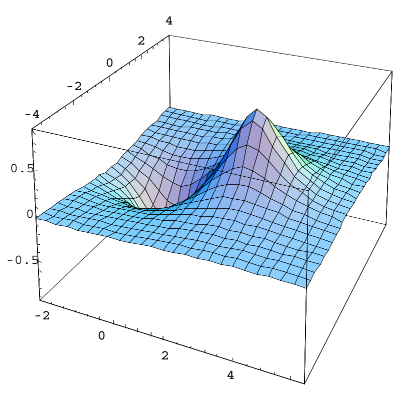
Gradient

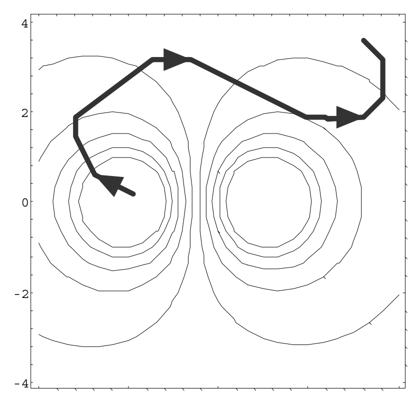
1. In MathLand, the temperature is a function of your coordinates (x,y): $T=100-x^2-y^2$. Some bug starts out at (a, b) and want to get warm as fast as possible. In which direction should the bug move? If the bug moves 1 unit / sec, how fast is he getting warmer every second?
2. The bug from problem 2 got an A+ in Math 21a and thus has a strategy for finding the warmest climate: constantly adjust her velocity to equal the gradient of the temperature. What is the bug's position as a function of time? Where does she end up?

Integration

1. Set up an integral to find the volume of a 1 mm thick cylindrical washer with an outer diameter of 30 mm and an inner diameter of 10 mm.
2. Set up an integral to find the volume of sphere with radius R with a cylindrical hole of radius A drilled down the middle.
3. The function $y=f(x)$ is rotated about the x-axis to form the surface of a solid. What is the volume of the solid between $x=a$ and $x=b$?

A 3D and a contour plot of $f(x, y)$ are shown to the right, with part of a constraint function $g(x,y,z)=0$ superimposed on the contour plot.





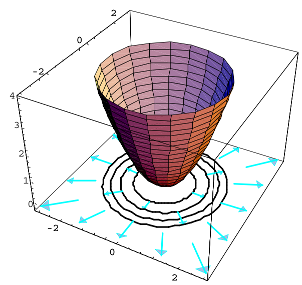
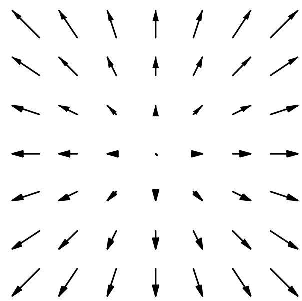
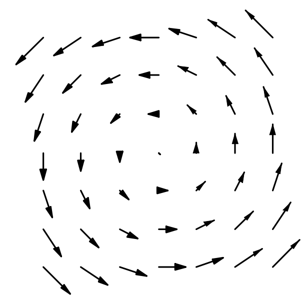
A. If you are moving in the direction indicated by arrows, when is the directional derivative of $f(x, y)$ along the constraint curve positive? Negative? Zero? How does this relate to the gradient?

B. Say you want to optimize $f(x, y)$ subject to the constraint. What point(s) on the graph satisfy the Lagrange condition: $\nabla f = \lambda \nabla g$?

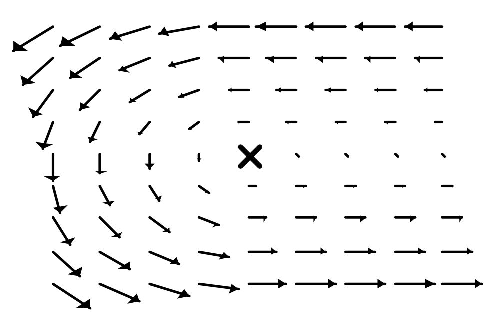
CA: Ed Boas

Week 9-10: Grad, Div, and Curl

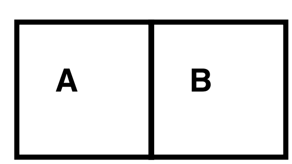
Summary

	grad	***div***	***curl***
Meaning	grad (scalar field) = vector field points “uphill” with magnitude of directional derivative in that direction	div (vector field) = scalar field flux / area (2D vector field) or flux / volume (3D vector field)	curl(2D vector field)=scalar fld. flow / area
Picture	 *Arrows are gradient of function whose graph and level curves are shown.*	 *Vector field with constant positive divergence*	 *Vector field with constant positive curl*
Formula	grad f $= \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$	div **F** $= \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$	curl $\mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
Integral theorems	flow of a conservative field (i.e. a gradient) along a path = difference in potential at endpoints $\int_a^b (\nabla f)\cdot d\mathbf{r} = f(b) - f(a)$	flux across closed curve (or surface) = area (or volume) integral of divergence *inside* the curve (or surface) $\oint \mathbf{F}\cdot\mathbf{n}\,ds = \iint \text{div}\,\mathbf{F}\,dA$ $\iint \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint \text{div}\,\mathbf{F}\,dV$	flow along closed curve = area integral of curl *inside* the curve $\oint \mathbf{F}\cdot d\mathbf{r} = \iint \text{curl}\,\mathbf{F}\,dA$
Other	Directional derivative: $\nabla f \cdot \vec{u}$ ∇f is a conservative field with potential f. ∇f is perpendicular to the level curves of f	The following statements are equivalent: 1. div **F** = 0 everywhere 2. No flux across any closed curve or surface	The following statements are equivalent: 1. curl **F** = 0 everywhere 2. **F** is conservative 3. No flow along any closed curve

Problems



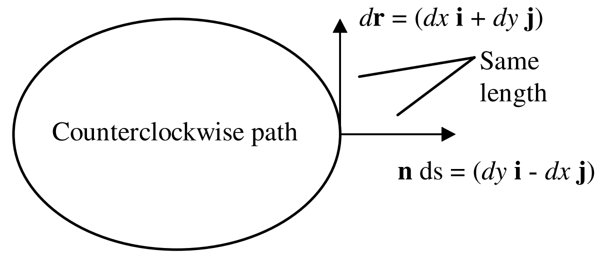
1. Is the vector field shown to the right conservative? Is the curl positive, negative, or zero at the ×? How about the divergence?
2. Simplify: curl grad f. [Hint: is grad f conservative?]
3. If the graph of $z = f(x, y)$ is concave down at the origin, then is div grad f positive, negative, or 0 at the origin? [Note: div grad f, also written $\nabla^2 f$, is called the Laplacian]
4. The figure to the right shows two regions, A and B, in a vector field (not shown). Express the flux across the boundary surrounding both A and B in terms of the flux across the boundary of A and the flux across the boundary of B. Does the same relationship hold for the flow? [Hint: think about the contribution of each line segment to the total flux or flow]



Week 9-10: Integrals in Vector Fields

Summary

- When evaluating flow and flux integrals, it is helpful to think of the components of the infinitesimal path segment: $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$
- **Flow**: How much of a vector field goes *along* a path.
  $\int \mathbf{F} \cdot d\mathbf{r} = \int (F_1\mathbf{i} + F_2\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \int F_1\,dx + F_2\,dy$
- **Flux**: How much of a vector field goes *across* a path.
  $\int \mathbf{F} \cdot \mathbf{n}ds = \int (F_1\mathbf{i} + F_2\mathbf{j}) \cdot (dy\mathbf{i} - dx\mathbf{j}) = \int F_1\,dy - F_2\,dx$



- **To evaluate flow and flux integrals**, parameterize the path, express dx, dy, and the vector field in terms of t (the parameter for the path), then perform the integral over t.
- **To perform integrals over surfaces**: $d\sigma = \dfrac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|}dA$, where

  $g\,(x, y, z) = c$ describes the surface
  $d\sigma$ is the infinitesimal area on the surface
  dA is the infinitesimal area in the base region
  $\mathbf{p}$ is a vector perpendicular to the base region
- The above relation converts integrals over surfaces (to find area, mass, center of mass, etc.) into integrals in the base region
- **Flux across a surface**: How much of a vector field goes *across* the surface $g\,(x, y, z) = c$:
  $$\iint \mathbf{F} \cdot \mathbf{n}d\sigma = \iint F \cdot \left(\pm\frac{\nabla g}{|\nabla g|}\right)\frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|}dA = \iint F \cdot \frac{\pm\nabla g}{|\nabla g \cdot \mathbf{p}|}dA$$
- **Other ways to evaluate integrals over vector fields** (besides direct computation)

  Flow
  A. For constant $\mathbf{F} \cdot d\mathbf{r}/|d\mathbf{r}|$, multiply $\mathbf{F} \cdot d\mathbf{r}/|d\mathbf{r}|$ by length of curve [Note: $d\mathbf{r}/|d\mathbf{r}|$ is a unit vector *parallel* to the curve]
  B. Integrate curl inside region (for a closed curve)
  C. Find a potential function then subtract the potential at the endpoints

  Flux
  A. For constant $\mathbf{F} \cdot \mathbf{n}$, multiply $\mathbf{F} \cdot \mathbf{n}$ by length of curve (2D field) or area of surface (3D field)
  B. Integrate divergence inside region (for a closed curve or surface)
  C. Add a conveniently chosen "cap" to close the surface, apply method (B), then subtract off the flux through the added "cap."

Problems

1. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across a unit sphere centered at the origin using three different methods:
   A. The "long" way (using the formula given above)
   B. Volume integral of the divergence
   C. (magnitude of field on the surface) $\times$ (area of surface)
2. Find the upwards flux of $\mathbf{F} = 2\,\mathbf{k}$ through the surface $z = x^2 - y^2 + 4$, $-1 < x < 1$, $-1 < y < 1$ using two different methods:
   A. The "long" way
   B. Choose a convenient closed surface containing the surface of this problem. Then, use the fact that the divergence of $\mathbf{F}$ is 0 to evaluate the flux.
3. Does the flux of $-y\,\mathbf{i} + x\,\mathbf{j}$ between two endpoints depend on the path taken? [Hint: set up the flux integral, reinterpret it as a flow integral, then try to find a potential function]
4. Review problem: Set up an integral to find the volume of a torus with the following cross section: